# Problem Set 7

## Macroeconomics III

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# Problem 1

We have the following:

- Stock price: $p_t$
- Dividends of the stock: $d_t$
- Riskless asset with a constant return of $r$

Risk neutral valuation or by a no-arbitrage condition implies:

$$p_t = \frac{\mathbb{E}[p_{t+1}|I_t] + d_t}{1+r} \tag{1}$$

The dividend follows the following process:

$$d_t = (1-\rho)d_0 + \rho d_{t-1} + \upsilon_t \tag{2}$$

Where $\upsilon_t \sim N(0, \sigma^2)$. The parameters are such that $0 < \rho < 1$ and $d_0 > 0$.

# Problem 1

The price is given by the following no-arbitrage condition:

$$\underbrace{p_t(1+r)}_{\text{Return on the riskless asset}} = \underbrace{\mathbb{E}[p_{t+1}|I_t] + d_t}_{\text{Expected return on the stock}}$$

The following ways of denoting expectation in period $t$ is equivalent:

$$\mathbb{E}[p_{t+1}|I_t] = \mathbb{E}_t[p_{t+1}]$$

**Useful results:**

The sum of a finite geometric series for $|r| < 1$ is given by:

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

The sum of an infinite geometric series for $|r| < 1$ is given by:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

# Problem 1a - Price

Solve for the price of the stock at time $t$ as a function of the current and past dividends. Explain.

$$p_t = \frac{\mathbb{E}[p_{t+1}|I_t] + d_t}{1+r}$$
$$d_t = (1-\rho)d_0 + \rho d_{t-1} + \upsilon_t$$

1. Find $p_t$ as a function of expected future dividends. Substitute recursively using the first equation above.
2. Find the expectation of future dividends. Substitute recursively using the second equation from above. Find an expression of $d_{t+n}$ based on $d_0$, $\rho$, $d_t$ and $\upsilon$. Take expectations - what happens to $\upsilon$?
3. Insert the expectations of dividends into the price function from 1.

Use the two following equations to rewrite the sums you encounter

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r} \qquad \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

# Problem 1a - Price - Future dividends (1/4)

Find $p_t$ as a function of the current and past dividends. Explain.

We find the price as a function of expected future dividends by recursive substitution of equation (1).[1]

$$
\begin{aligned}
p_t &= \frac{\mathbb{E}_t[p_{t+1}] + d_t}{1+r} \\
&= \frac{d_t}{1+r} + \frac{\mathbb{E}_t[p_{t+2}] + \mathbb{E}_t[d_{t+1}]}{(1+r)^2} \\
&= \frac{d_t}{1+r} + \frac{\mathbb{E}_t[d_{t+1}]}{(1+r)^2} + \frac{\mathbb{E}_t[p_{t+3}] + \mathbb{E}_t[d_{t+2}]}{(1+r)^3} \\
&= \sum_{i=0}^{n-1} \frac{\mathbb{E}_t[d_{t+i}]}{(1+r)^{i+1}} + \underbrace{\lim_{n\to\infty}\left(\frac{\mathbb{E}_t[p_{t+n}]}{(1+r)^n}\right)}_{=0 \text{ By assumption}} \\
&\xrightarrow[\text{for } n\to\infty]{} \sum_{i=0}^{\infty} \frac{\mathbb{E}_t[d_{t+i}]}{(1+r)^{i+1}}
\end{aligned}
$$

[1] Law of iterated expectations: $\mathbb{E}_t[d_{t+1}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}[d_{t+1}]\right]$.

# Problem 1a - Future dividends (2/4)

We now have the price as a function of expected future dividends.

$$p_t = \sum_{i=0}^{\infty} \frac{\mathbb{E}_t[d_{t+i}]}{(1+r)^{i+1}}$$

To find the expectation of future dividends we use the process of the dividends.

$$\begin{aligned}
d_{t+1} &= (1-\rho)d_0 + \rho d_t + \upsilon_{t+1} \\
d_{t+2} &= (1-\rho)d_0 + \rho d_{t+1} + \upsilon_{t+2} \\
&= (1-\rho)d_0 + \rho[(1-\rho)d_0 + \rho d_t + \upsilon_{t+1}] + \upsilon_{t+2} \\
&= (1-\rho)d_0(1+\rho) + \rho^2 d_t + \upsilon_{t+2} + \rho\upsilon_{t+1} \\
d_{t+3} &= (1-\rho)d_0 + \rho d_{t+2} + \upsilon_{t+3} \\
&= (1-\rho)d_0(1+\rho+\rho^2) + \rho^3 d_t + \upsilon_{t+3} + \rho\upsilon_{t+2} + \rho^2\upsilon_{t+1}
\end{aligned}$$

# Problem 1a - Future dividends (3/4)

Iterating $n$ periods forward yields:

$$d_{t+n} = (1-\rho)d_0 \sum_{i=0}^{n-1} \rho^i + \rho^n d_t + \sum_{i=0}^{n-1} \rho^i \upsilon_{t+n-i} \tag{3}$$

We take the expectation of the sum as that is what we're looking for.

$$\mathbb{E}_t[d_{t+n}] = (1-\rho)d_0 \sum_{i=0}^{n-1} \rho^i + \rho^n d_t \tag{4}$$

Since $\mathbb{E}_t[\upsilon_{t+i}] = 0$ for $i > 0$. Next, find the sum of the geometric series:

$$(1-\rho)d_0 \sum_{i=0}^{n-1} \rho^i = (1-\rho)d_0 \frac{1-\rho^n}{1-\rho} = d_0(1-\rho^n)$$

Hence, we get:

$$\begin{aligned} \mathbb{E}_t[d_{t+n}] &= d_0(1-\rho^n) + \rho^n d_t \\ &= d_0 + \rho^n(d_t - d_0) \end{aligned}$$

Alternative method

# Problem 1a - Price of the stock (4/4)

We insert the expectations of the dividends into the stock price:

$$
\begin{aligned}
p_t &= \sum_{i=0}^{\infty} \frac{\mathbb{E}_t[d_{t+i}]}{(1+r)^{i+1}} \\
&= \sum_{i=0}^{\infty} \frac{d_0 + \rho^i (d_t - d_0)}{(1+r)^{i+1}} \\
&= d_0 \sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} + (d_t - d_0) \sum_{i=0}^{\infty} \frac{\rho^i}{(1+r)^{i+1}}
\end{aligned}
$$

Next we find the geometric sums (if unclear, see next slide):

$$
\sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} = \frac{1}{r} \qquad \text{and} \qquad \sum_{i=0}^{\infty} \frac{\rho^i}{(1+r)^{i+1}} = \frac{1}{1+r-\rho}
$$

Inserting into the price yields:

$$
p_t = \frac{d_0}{r} + \frac{d_t - d_0}{1+r-\rho}
$$

# Problem 1a - Deriving the geometric sums (extra)

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{\rho^i}{(1+r)^{i+1}} &= \frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{\rho}{(1+r)}\right)^i \\ &= \frac{1}{1+r} \frac{1}{1-\frac{\rho}{1+r}} \\ &= \frac{1}{1+r} \frac{1+r}{1+r-\rho} \\ &= \frac{1}{1+r-\rho}\end{aligned}$$

We see that for the special case where $\rho = 1$ it becomes

$$\sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} = \frac{1}{r} \tag{5}$$

# Problem 1b - Variance of the stock price (1/2)

Calculate the unconditional variance of the stock price as a function of $\sigma^2$ and other parameters.

The price of the stock is:

$$p_t = \frac{d_0}{r} + \frac{d_t - d_0}{1 + r - \rho}$$

We know that $Var(d_0) = 0$. Hence, the variance of the price is:

$$Var(p_t) = \frac{1}{(1 + r - \rho)^2} Var(d_t)$$

We then turn to the variance of $d_t$.

$$d_t = (1 - \rho)d_0 + \rho d_{t-1} + \upsilon_t$$

Substitution recursively yields:

$$d_t = (1 - \rho)d_0(1 + \rho + \rho^2 + \ldots + \rho^{t-1}) + \upsilon_t + \rho\upsilon_{t-1} + \ldots + \rho^{t-1}\upsilon_1$$

# Problem 1b - Variance of stock price (2/2)

The variance of $d_t$ is:

$$Var(d_t) = (1 + \rho^2 + \rho^4 + \ldots + \rho^{2(t-1)})\sigma^2$$

Since $Cov(\upsilon_t, \upsilon_{t+1}) = 0$ and $Var(\rho\upsilon_t) = \rho^2 Var(\upsilon_t) = \rho^2\sigma^2$.

Rewriting the variance as it is a geometric sum yields:

$$\begin{aligned} Var(d_t) &= (1 + \rho^2 + \rho^4 + \ldots + \rho^{2(t-1)})\sigma^2 = \sum_{i=0}^{t-1} \rho^{2i}\sigma^2 \\ &= \frac{1-\rho^{2t}}{1-\rho^2}\sigma^2 \end{aligned}$$

Inserting into the variance of the price yields:

$$\begin{aligned} Var(p_t) &= \frac{1}{(1+r-\rho)^2} Var(d_t) \\ &= \frac{1}{(1+r-\rho)^2} \frac{1-\rho^{2t}}{1-\rho^2}\sigma^2 \end{aligned}$$

Alternative method

# Problem 1c - How does the variance change if $\rho \uparrow$

How does the variance of the stock price change as $\rho$ increases?

The answer is relatively straightforward when looking at the variance of $p_t$ and $d_t$.

$$Var(p_t) = \frac{1}{(1+r-\rho)^2} Var(d_t)$$
$$Var(d_t) = (1 + \rho^2 + \rho^4 + \ldots + \rho^{2(t-1)})\sigma^2$$

- $Var(p_t)$: The fraction $\frac{1}{(1+r-\rho)^2}$ clearly increases as $\rho$ increases since $0 < \rho < 1$. It makes the denominator smaller and thus the fraction greater.
- $Var(d_t)$: Clearly increases when the persistence of shocks increase.

Overall, an increase in $\rho$ leads to an increase in the unconditional variance of $p_t$.

# Problem 1d - Show that $Var(p_t') > Var(p_t)$ (1/2)

Show that the variance of $p_t'$ is larger than the variance of $p_t$ (this relates to $p$ being a forecast of $p'$ and thus having lower variance).

The two prices are given by:

$$p_t = \sum_{i=0}^{\infty} \frac{\mathbb{E}_t[d_{t+i}]}{(1+r)^{i+1}}$$

$$p_t' = \sum_{i=0}^{\infty} \frac{d_{t+i}}{(1+r)^{i+1}}$$

Actual and expected dividends are given by equations (3) and (4):

$$d_{t+n} = (1-\rho)d_0 \sum_{i=0}^{n-1} \rho^i + \rho^n d_t + \sum_{i=0}^{n-1} \rho^i \upsilon_{t+n-i}$$

$$\mathbb{E}_t[d_{t+n}] = (1-\rho)d_0 \sum_{i=0}^{n-1} \rho^i + \rho^n d_t$$

# Problem 1d - Show that $Var(p'_t) > Var(p_t)$ (2/2)

Hence, the actual and expected dividends are connected in the following manner

$$d_{t+n} = \mathbb{E}_t[d_{t+n}] + \underbrace{\sum_{i=0}^{n-1} \rho^i \upsilon_{t+n-i}}_{=f_t(\upsilon_n)}$$

The ex post price and the original price is, therefore:

$$p'_t = p_t + \sum_{i=0}^{\infty} \frac{f_t(\upsilon_i)}{(1+r)^{i+1}}$$

From this, it is evident that the ex-post price has a higher variance than the ex-ante price. The forecast of the price, $p_t$ does not include the shocks as they are all zero in expectation. The actual price does, however, include it.

# Problem 1e - Variance when $\rho = 1$ (1/2)

Suppose now that $\rho = 1$, with the dividend following a random walk. What happens to the unconditional variance of the price of the stock?

The expectations of the dividends are now such that:

$$\mathbb{E}_t[d_{t+1}] = \mathbb{E}_t[d_t + \upsilon_{t+1}] = d_t$$

The price is given by:

$$\begin{aligned} p_t =& \frac{\mathbb{E}_t[d_t] + d_t}{1+r} = \frac{d_t}{1+r} + \frac{\mathbb{E}_t[p_{t+2} + d_{t+1}]}{(1+r)^2} \\ & \dots \\ =& \sum_{i=0}^{\infty} \frac{d_t}{(1+r)^{i+1}} = \frac{d_t}{r} \end{aligned}$$

The variance of the price is, therefore:

$$Var(p_t) = \frac{Var(d_t)}{r^2}$$

# Problem 1e - Variance when $\rho = 1$ (2/2)

We turn to the variance of the dividends.

$$\begin{aligned} d_t =& d_{t-1} + \upsilon_t = d_{t-2} + \upsilon_{t-1} + \upsilon_t \\ & \dots \\ d_t =& d_0 + \upsilon_1 + \upsilon_2 + \ldots + \upsilon_{t-1} + \upsilon_t \end{aligned}$$

Hence, the variance of the dividends at time $t$ is:

$$Var(d_t) = t\sigma^2$$

The variance of the price thus becomes:

$$Var(p_t) = \frac{t\sigma^2}{r}$$

We see that the variance is increasing in $t$ and $Var(p_t) \to \infty$ for $t \to \infty$. The unconditional variance keeps increasing as the dividends follow a random walk.

# Problem 2

We have the following model:

$$p^f = (1-\phi)p + \phi m \tag{6}$$
$$p^r = (1-\phi)\mathbb{E}[p] + \phi\mathbb{E}[m] \tag{7}$$
$$p = qp^r + (1-q)p^f \tag{8}$$
$$y = m - p \tag{9}$$

$p^f$ is the price of the flexible firms, $p^r$ price of the rigid firms, $p$ overall price level, $q$ share of rigid firms and $y$ is the output.[a]

The timing is:

1. Rigid price-level firms set their prices equal to the expected optimal prices: $\mathbb{E}[p^f] = p^r$.
2. Money supply $m$ is determined.
3. Flexible price-level firms set their optimal prices such that $p$ and $y$ is realised.

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[a] Log-version of $M = PY$, money supply equals money demand.

# Problem 2a - Price of the flexible firms

Find $p^f$ in terms of $p^r$, $m$ and the parameter of the model ($\phi$ and $q$)

Plug the price level into the flexible prices:

$$\begin{aligned}
p^f &= (1-\phi)\big(qp^r + (1-q)p^f\big) + \phi m \\
p^f\big(1-(1-\phi)(1-q)\big) &= (1-\phi)qp^r + \phi m \\
p^f\big(\phi + (1-\phi)q\big) &= (1-\phi)qp^r + \phi m - \phi p^r + \phi p^r \\
p^f\big(\phi + (1-\phi)q\big) &= \big(\phi + (1-\phi)q\big)p^r + \phi(m-p^r) \\
p^f &= p^r + \frac{\phi}{\phi + (1-\phi)q}(m-p^r)
\end{aligned}$$

Thus, we have an expression for the flexible prices based on the $\phi$, $q$, $m$ and $p^r$.

# Problem 2b - Price of the rigid firms (1/2)

Find $p^r$ in terms of $\mathbb{E}[pm]$ and the parameters of the model.

Plug the expression for $p^f$ into the price level in order to get rid of $p^f$:

$$p = qp^r + (1-q)\underbrace{\left(p^r + \frac{\phi}{\phi + (1-\phi)q}(m - p^r)\right)}_{p^f}$$

$$p = p^r + \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - p^r)$$

Plug this into the expression for rigid prices and use that they know their own prices, $\mathbb{E}[p^r] = p^r$:

$$p^r = (1-\phi)\mathbb{E}[p] + \phi\mathbb{E}[m]$$

$$p^r = (1-\phi)\mathbb{E}\left[p^r + \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - p^r)\right] + \phi\mathbb{E}[m]$$

$$p^r = (1-\phi)p^r + \frac{(1-q)\phi}{\phi + (1-\phi)q}(\mathbb{E}[m] - p^r) + \phi\mathbb{E}[m]$$

# Problem 2b - Price of the rigid firms (2/2)

We know that the price is given by

$$
\begin{aligned}
p^r &= (1-\phi)p^r + \frac{(1-q)\phi}{\phi+(1-\phi)q}(\mathbb{E}[m] - p^r)] + \phi\mathbb{E}[m] \\
0 &= \frac{(1-q)\phi}{\phi+(1-\phi)q}(\mathbb{E}[m] - p^r) - \phi p^r + \phi\mathbb{E}[m] \\
0 &= \frac{(1-q)\phi}{\phi+(1-\phi)q}(\mathbb{E}[m] - p^r) + \phi(\mathbb{E}[m] - p^r) \\
0 &= \Big[\frac{(1-q)\phi}{\phi+(1-\phi)q} + \phi\Big](\mathbb{E}[m] - p^r)]
\end{aligned}
$$

For this to be equal to zero, the last parentheses must be zero,

$$p^r = \mathbb{E}[m]$$

# Problem 2c - Effect of expected change in $m$ (1/2)

Do anticipated changes in $m$ (i.e. changes known at the time rigid price firms set their prices) affect $y$? Why or why not?

We insert the rigid firm prices into the aggregate level prices:

$$p = p^r + \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - p^r)$$
$$p = \mathbb{E}[m] + \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - \mathbb{E}[m])$$

We insert the prices into the output expression

$$\begin{aligned} y &= m - p \\ &= m - \mathbb{E}[m] - \frac{(1-q)\phi}{\phi + (1-\phi)q}(m - \mathbb{E}[m]) \\ &= \left(1 - \frac{(1-q)\phi}{\phi + (1-\phi)q}\right)(m - \mathbb{E}[m]) \end{aligned}$$

# Problem 2c - Effect of expected change in $m$ (2/2)

Continuing the derivations, we know that the following holds:

$$\begin{aligned} y &= \left(1 - \frac{(1-q)\phi}{\phi + (1-\phi)q}\right)(m - \mathbb{E}[m]) \\ &= \frac{\phi + (1-\phi)q - (1-q)\phi}{\phi + (1-\phi)q}(m - \mathbb{E}[m]) \\ &= \frac{q}{\phi + (1-\phi)q}(m - \mathbb{E}[m]) \end{aligned}$$

For an expected change in $m$ in holds that $\mathbb{E}[m] = m$. Hence, an expected change in $m$ does not change the output.

Intuitively, increasing the money supply only affects the price level if the increase is expected.

# Problem 2d - Effect of unexpected change in $m$

Do unanticipated changes in $m$ affect $y$? Why or why not?

$$y = \frac{q}{\phi + (1-\phi)q}(m - \mathbb{E}[m])$$

For an unanticipated change in $m$ in holds that $\mathbb{E}[m] \neq m$.

An unanticipated increase in $m$ leads to an increase in $y$ since $m > \mathbb{E}[m]$, which makes the last parentheses positive. The sudden increase in $m$ can be seen as a short term increase in demand. Since only the flexible firms, can adopt the correct prices, the price level will be lower than optimally, which results in a higher demand.

The effect is given by:

$$y_m = \frac{\partial y}{\partial m} = \frac{q}{\phi + (1-\phi)q}$$

# Problem 2e - Extra question - Importance of $q$ and $\phi$

**How does $q$ and $\phi$ affect $y_m$?**

$$\frac{\partial y_m}{\partial q} = \frac{\phi}{\left(\phi + (1-\phi)q\right)^2} > 0$$

**Intuition:** If there are more rigid price firms, less firms will be able to change their prices. Hence, the aggregate price level will be relatively more rigid. In a case where there is an unanticipated increase in $m$, the aggregate-price level will be lower for higher $q$.

$$\frac{\partial y_m}{\partial \phi} = \frac{-q(1-q)}{\left(\phi + (1-\phi)q\right)^2} < 0$$

**Intuition:** $\phi$ determines the extend to which the firms will try to match the money supply. The rigid prices won't be affected but the flexible firms will set their prices closer to $m$ for higher $\phi$. Hence, the effect of a change in $m$ will be lower for a high $\phi$.

# Problem 1a - Future dividends - Alternative (1/2)

Next, we want to find an expression of the expectation of future dividends based on information available at time $t$.

From eq. (2) we know that the dividends in period $t+k$ are:

$$\begin{aligned} d_{t+i} &= (1-\rho)d_0 + \rho d_{t+i-1} + \upsilon_{t+i} \\ d_{t+i} - d_0 &= \rho(d_{t+i-1} - d_0) + \upsilon_{t+i} \end{aligned}$$

The expected difference between the dividends in period $t+j+k$ and 0 based on information available in period $t$ is:

$$\begin{aligned} \mathbb{E}_t[d_{t+i} - d_0] &= \mathbb{E}_t[\rho(d_{t+i-1} - d_0) + \upsilon_{t+i}] \\ &= \rho\mathbb{E}_t[d_{t+i-1} - d_0] \end{aligned} \tag{10}$$

Since $\mathbb{E}_t[\upsilon_{t+k}] = 0$. Equivalently the expectation for period $t+k-1$ is:

$$\mathbb{E}_t[d_{t+i-1} - d_0] = \rho\mathbb{E}_t[d_{t+i-2} - d_0] \tag{11}$$

# Problem 1a - Future dividends - Alternative (2/2)

We combine equations (10) and (11).

$$\mathbb{E}_t[d_{t+i} - d_0] = \rho^2 \mathbb{E}_t[d_{t+i-2} - d_0]$$

Iterating until we have an expression based on information available at time $t$.

$$\mathbb{E}_t[d_{t+i} - d_0] = \rho^k \mathbb{E}_t[d_t - d_0]$$

**Note:** $\mathbb{E}_t[d_t] = d_t$ and $\mathbb{E}_t[d_0] = d_0$.

$$\begin{aligned} \mathbb{E}_t[d_{t+i}] - d_0 &= \rho^i (d_t - d_0) \\ \mathbb{E}_t[d_{t+i}] &= d_0 + \rho^i (d_t - d_0) \end{aligned}$$

Hence, we know have the expectations of future dividends based on information available at time $t$.

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# Problem 1b - Variance - Alternative

$$
\begin{aligned}
d_t &= (1-\rho)d_0 + \rho d_{t-1} + \upsilon_t \\
Var(d_t) &= \rho^2 Var(d_{t-1}) + \sigma^2 \\
Var(d_t) &= \rho^2 \underbrace{(\rho^2 Var(d_{t-2}) + \sigma^2)}_{=Var(d_{t-1})} + \sigma^2 \\
Var(d_t) &= \rho^4(\rho^2 Var(d_{t-3} + \sigma^2) + (1+\rho^2)\sigma^2 \\
&\dots \\
&= \rho^{2t} Var(d_0) + \sum_{i=0}^{t-1} \rho^{2i}\sigma^2 \\
&= \frac{1-\rho^2 t}{1-\rho^2}\sigma^2
\end{aligned}
$$

Inserting into the variance of the price yields:

$$
var(p_t) = \frac{1}{(1+r-\rho)^2} \frac{1-\rho^{2t}}{1-\rho}\sigma^2
$$

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